# Technical Notes

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## Continuous Stress Fields in Finite Element Analysis

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### Introduction

THIS paper concerns classical finite-element analysis of elasticity problems. An algorithm is described to obtain a continuous stress field as well as the usual continuous displacement field. Numerical experimentation with various problems show a distinct improvement of solutions obtained with coarse mesh systems.

### Fundamental Equations

The fundamental equation of elastostatics used in the finite-element method lead to[1]

$$[K]\{\delta\}+\sum_{\text{Elem.}}\int_{\text{Vol.}}[B]^T\{\sigma\}\text{d(Vol)}-\sum_{\text{Elem.}}\int_{\text{Vol.}}[B]^T[D]\{\epsilon\}\text{d(Vol)}=\{R\} \tag{1}$$

where $[K]$ is the stiffness matrix; $[B]$ is the transformation matrix between strains and nodal displacements; $\{\sigma\}$, $\{\epsilon\}$ are vectors of initial stresses and strains at the element level; $[D]$ is the matrix of elastic properties in the constitutive law; $\{R\}$ is the vector of nodal forces. From Ref. 1 strains and stresses are written as

$$\{\epsilon\}=[B]\{\delta\} \tag{2}$$

$$\{\sigma\}=[D]\{\epsilon\}=[D][B]\{\delta\} \tag{3}$$

The strains and stresses so obtained are discontinuous at the nodes for a given mesh, but the discontinuities decrease with mesh refinement.[2]

### Stress Field

For a structure in which a continuous stress field is known to exist, the discontinuous results obtained from the finite-element method, element by element at a node, can be used to obtain a set of averaged nodal stresses. Various procedures have been used to obtain the average nodal stresses[3] and in the present context any such procedure can be employed. For the problems presented below, a simple average of all results is taken for every node. With a set of nodal values for the stresses it is then easy to construct a continuous stress field over the domain of definition of the problem. In the problems presented in the following the same shape functions as those used for the displacement are employed.

$$\{\sigma_c\}=[N]\{\sigma_j\} \tag{4}$$

where $[N]$ is a set of shape functions associated with the element used, $\{\sigma_j\}$ is a vector of nodal stresses obtained by an averaging technique, $\{\sigma_c\}$ is the continuous stress field at the element level.

The continuous stress field of Eq. (4) differs from the calculated stress field of Eq. (3) because of the discretization error, but the difference would have to decrease by mesh refinement since the original solution is known to converge to an exact result by such a process, i.e.,

$$\lim_{n\to\infty}(\{\sigma_c\}-\{\sigma\})=0 \tag{5}$$

where $n$ is the total number of elements in the mesh.

### Loubignac Iterations

If the difference between stress fields is treated as a set of initial stresses, the original problem can be reformulated iteratively as

$$[K]\{\delta^{(i)}\}+\sum_{\text{Elem.}}\int_{\text{Vol.}}[B]^T(\{\sigma_c^{(i-1)}\}-\{\sigma^{(i-1)}\})\text{d(Vol)}=\{R\} \tag{6}$$

Using Eqs. (4) and (3) in Eq. (6) gives after reduction and regrouping of terms:

$$[K](\{\delta^{(i)}\}-\{\delta^{(i-1)}\})=\{R\}-\sum_{\text{Elem.}}\int_{\text{Vol.}}[B]^T[N]\{\sigma_j^{(i-1)}\}\text{d(Vol)} \tag{7}$$

Equation (7) is solved iteratively until $(\{\delta^{(i)}\}-\{\delta^{(i-1)}\})$ is sufficiently small. The last values of $\{\delta\}$ and $\{\sigma_j\}$ are retained as solutions. The first values of nodal stresses are obtained from

$$[K]\{\delta^{(o)}\}=\{R\} \tag{8}$$

$$\{\sigma_j^{(o)}\}=\frac{1}{m}\sum_{e=1}^{m}\{\sigma_j^e\} \tag{9}$$

In Eq. (9), the summation is for all the elements that share node $j$. Very little effort is needed to iterate Eq. (7) since Eq. (8) has already been solved. The most time consuming part of the iteration is the formation of the initial stress load vector; obtaining the value of $(\{\delta^{(i)}\}-\{\delta^{(i-1)}\})$ involves only forward and backward substitution into the already deflated system.

### Numerical Examples

The first problem defined in Fig. 1 is taken from Ref. 4. The cantilevered beam was solved using constant strain triangles. Five different meshes were used and results for the tip deflection at $x=48$ and $y=6$, as well as the flexural stress at $x=12$ and $y=12$, are plotted in Figs. 2 and 3. For every solution, five Loubignac iterations were performed and the figures show the results with and without iterations, as well as analytical results. The tip deflections are too large but the

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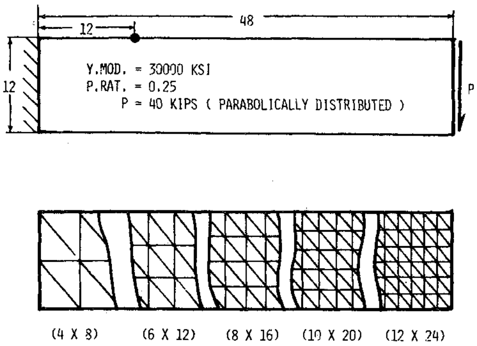

Fig. 1 End-loaded cantilevered beam.

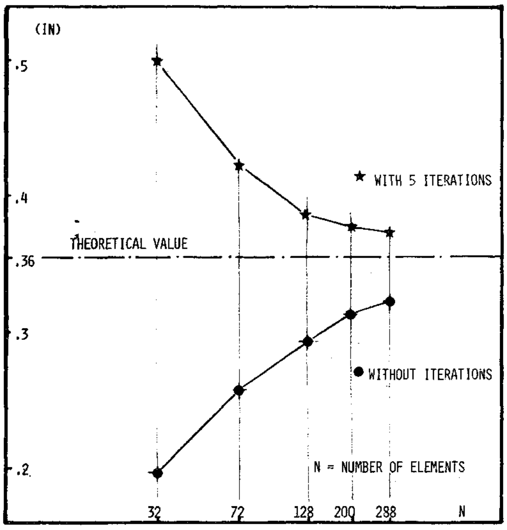

Fig. 2 Tip deflection of cantilevered beam.

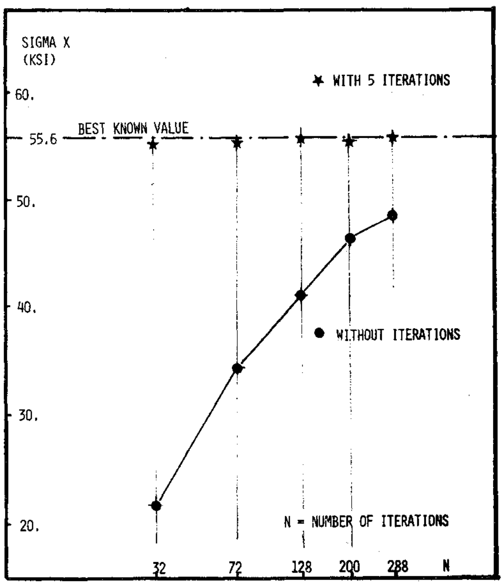

Fig. 3 Stess at point A (see Fig. 1).

tendency to converge to the analytical result by mesh refinement is clear. The stress values, however, are much better with iterations than without. Even the very coarse mesh gives excellent results.

The last problem is that of an axisymmetric thick cylinder internally loaded with a uniform pressure. The problem has a trivial solution but is of a type that normally gives bad results because stress boundary conditions cannot be imposed even though they are known a priori. The Loubignac iteration allows stress boundary conditions to be imposed very easily when the continuous stress field is constructed in Eq. (4). Figure 4 shows analytic stress contours vs results produced automatically with a finite-element system without iterations. In Fig. 5, the same analytic stress contours are shown vs results produced by the same finite element system modified to perform five Loubignac iterations. Figures 4 and 5 show the radial stress component everywhere. Again the results speak for themselves.

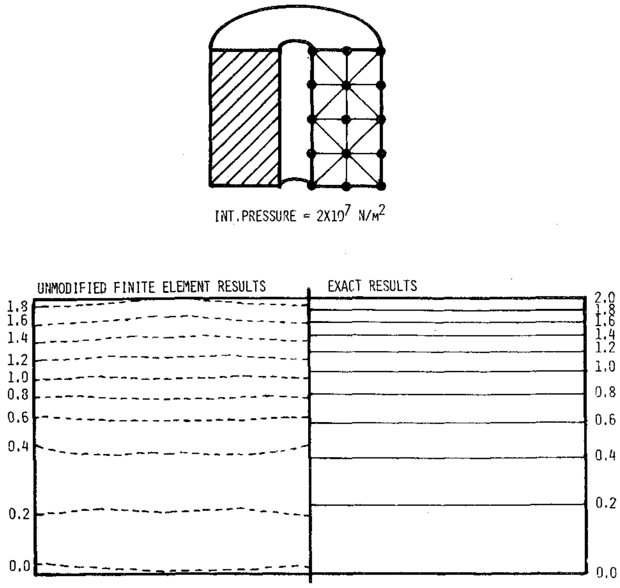

Fig. 4 Radial stress in internally pressurized cylinder.

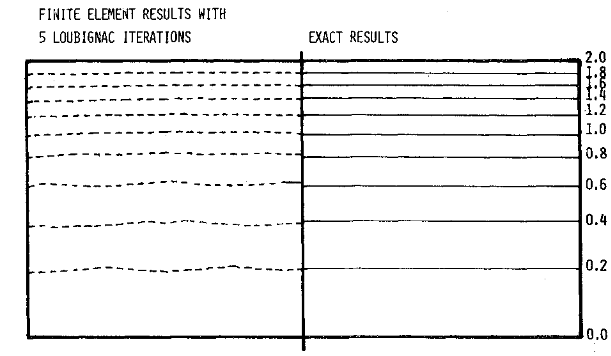

Fig. 5 Radial stress with Loubignac iterations.

## Conclusions

The algorithm presented in this communication is perfectly general and in all the problems solved so far with it, a distinct improvement of solutions has been obtained. All the computations have been performed at the computer center of the Centre Technique des Industries Mécaniques, Senlis, France.

The procedure described herein also allows stress boundary conditions to be imposed directly in a most straightforward manner and in that sense improves greatly upon the regular finite-element theory.

Obviously for an interface between two dissimilar materials, stresses can be discontinuous and generally unknown, the procedure described herein must evidently not be applied at such nodes. In practice, the algorithm can be coded with no loss of generality in a given system and offered as an option to a potential user.

## References

[1] Zienkiewicz, O.C., "The Finite Element Method in Engineering Science," Chap. 2, McGraw-Hill, New York, 1971.

[2]Pian, T.H.H. and Tong, P., "Basis of Finite Element Methods for Solid Continua," *International Journal for Numerical Methods in Engineering,* Vol. 1, 1969, pp. 3-28.
[3]Wilson, E.L., "Finite Element Analysis of Two Dimensional Structures," SESM 63-2, University of California at Berkeley, Berkeley, Calif., 1963.
[4]Felippa, C., "Refined Finite Element Analysis of Linear and Nonlinear Two Dimensional Structures," SESM 66-22, University of California at Berkeley, Berkeley, Calif., 1966.

# Scaling a Discrete Structural Model to Match Measured Modal Frequencies

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## Introduction

AMONG the various possibilities for schemes of its kind, this study deals with the specific case where the computer model of a structure is to be upgraded on the basis of experimental measurements of normal mode frequencies only. It is implicit that the essential forms of the fundamental modes are already adequately identified in the starting computer model. The objective is to establish a rational procedure for predicting the modified structure (stiffness matrix) which correlates most closely with a given set of measured modal frequencies. The form of the computer model itself (conforming elements) is taken to be given and fixed.

An existing procedure for this identification problem, as developed and used in the Viking program, apparently originates with White.[1] Experience with the structural modeling methods used in this program are summarized in a report of Wada et al.[2] Subsequent application of such procedures in the Skylab project is described in a recent report.[3] White's derivation establishes his model modification scheme based on the first-order terms of a Taylor expansion of the altered model about the original one. The scheme allows prediction of a set of scale parameters, in number equal to or less than the number of available test-data points. In the latter case the parameters are determined to minimize the mean-square difference between measured and computed frequencies.

There is no room to improve on White's result within the framework of his analytical derivation. However, by taking a different approach, a more broadly applicable procedure becomes available. Its advantages are that 1) both the number and choice of structural parameters to be varied in the modification are arbitrary, and 2) the possibility is provided that a given revision of the computer model may be approached through a set of lower-order problems with (generally) improved efficiency. The present approach relates to modal energies; its derivation leads to a useful boundedness property for iterate steps toward the upgraded computer model.

## Scaling Problem

For the linear eigenvalue problem, modal characteristics $\phi_j$ and $\lambda_j$ of the $j$th mode are associated with the equation

$$([K]-\lambda_j[M])\{\phi_j\}=\{0\} \tag{1}$$

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where $\lambda_j$ represents the square of the modal frequencies. System stiffness $[K]$ and mass $[M]$ are given in terms of element measures $k_i$ and $m_i$ by

$$[K]=\sum_{i=1}^{S}[R_i]^T[k_i][R_i] \tag{2}$$

and

$$[M]=\sum_{i=1}^{S}[T_i]^T[m_i][T_i] \tag{3}$$

for the structural system with $S$ elements. The associated functional, say $E_j$, is defined:†

$$E_j(K,\lambda_j,\phi_j^*)=\{\phi_j^*\}^T[K]\{\phi_j^*\}-\lambda_j\{\phi_j^*\}^T[M]\{\phi_j^*\} \tag{4}$$

Here $\phi_j^*$ represents any kinematically admissible field.

The actual eigenvectors $\phi_j$ are identified with stationary points (local minima) of $E_j(K, \lambda_j, \phi_j^*)$. Also for the actual eigenvalue

$$E_j(K,\lambda_j,\phi_j)=0 \tag{5}$$

Adopting the normalization

$$\{\phi_j^*\}^T[M]\{\phi_j^*\}=1 \tag{6}$$

Eq. (4) provides

$$\lambda_j=\{\phi_j\}^T[K]\{\phi_j\} \tag{7}$$

Only a linear scaling is considered in the present development (this corresponds to what is done in prior treatments). That is, the modification of structure, associated with a given application of the procedure, is a linear function of a set of scale parameters $\delta_i$. The structure $[K]$ obtained as the modification of starting structure $[K_0]$ is given by

$$[K]=[K_0]+\sum_{i=1}^{S}\delta_i[R_i]^T[k_i][R_i] \tag{8}$$

(In application the number of scale factors will generally be much less than $S$; in other words the $\delta_i$ would not all be independent.) Symbol $[\Delta]$ is introduced to represent the modification,

$$[\Delta]=\sum_{i=1}^{S}\delta_i[R_i]^T[k_i][R_i] \tag{9}$$

Equations (8) and (9) are used to substitute for $[K_0]$ in the expression for $E_j(K_0,\lambda_j^0,\phi_j^0)$. That is

$$\begin{aligned}E_j(K_0,\lambda_j^0,\phi_j^0)&=\{\phi_j^0\}^T[K_0]\{\phi_j^0\}-\lambda_j^0\\&=\{\phi_j^0\}^T[K]\{\phi_j^0\}-\{\phi_j^0\}^T[\Delta]\{\phi_j^0\}-\lambda_j^0=0\end{aligned} \tag{10}$$

Let the modification represent the difference between starting structure $[K_0]$ and modified structure $[K_e]$ with (test data) frequencies $\lambda_j^e$. With the introduction of this notation and the addition and subtraction of $\lambda_j^e$, Eq. (10) becomes

$$\{\phi_j^0\}^T[K_e]\{\phi_j^0\}-\lambda_j^e-\{\phi_j^0\}^T[\Delta_e]\{\phi_j^0\}+\lambda_j^e-\lambda_j^0=0$$

or

$$\begin{aligned}\{\phi_j^0\}^T[\Delta_e]\{\phi_j^0\}&=\lambda_j^e-\lambda_j^0+\{\phi_j^0\}^T[K_e]\{\phi_j^0\}-\lambda_j^e\\&=\lambda_j^e-\lambda_j^0+E_j(K_e,\lambda_j^e,\phi_j^0)\end{aligned} \tag{11}$$

†Only modification of stiffness is to be represented.